Commun. Math. Phys. 110, 215–236 (1987)

Communications in Mathematical Physics

The Shape Resonance

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Abstract. For a class of Schrödinger operators $H := -(\hbar^2/2m)\Delta + V$ on $L^2(\mathbb{R}^n)$, with potentials having minima embedded in the continuum of the spectrum and non-trapping tails, we show the existence of shape resonances exponentially close to the real axis as $\hbar \searrow 0$. The resonant energies are given by a convergent perturbation expansion in powers of a parameter exhibiting the expected exponentially small behaviour for tunneling.

I. Introduction

The concept of shape resonance has been introduced in the early days of quantum mechanics to resolve the puzzle of alpha-decay [Ga, GuGo]. As in the case of tunneling, the configuration space of the particle with energy ε in a potential V contains a region $J(\varepsilon) := \{x \in \mathbb{R}^n, V(x) > \varepsilon\}$ which is classically non-accessible and which for some values of ε, separates $\mathbb{R}^n$ into an exterior and interior region. The interior region stands for the nucleus, where the particle would be confined if it were not for the quantum mechanical tunneling through the barrier $J(\varepsilon)$ into the exterior. In the case of shape resonance the exterior extends typically to infinity (see Fig. 1a and b below).

In the case of tunneling and in particular in the case of shape resonance one is interested in situations where barrier penetration is small. This is expected to hold in the semiclassical regime: $k^2 := \hbar/(2m)^{1/2}$ small compared to $d(\mathbf{C}, (\partial J)_{\text{ext}})$ which denotes the Agmon distance between the exterior part $(\partial J(\varepsilon))_{\text{ext}}$ of $\partial J(\varepsilon)$ and the set $\mathbf{C}$ of points in the interior where V takes its minimal value v_0; d is derived from the metric $(ds)^2 := \max(0, V(x) - v_0)\, dx^2$.

In this introduction we shall describe the ideas of our analysis of shape resonances without going into precise technical definitions of the model (Sect. II).

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