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Singular Measures Without Restrictive Intervals*

Stewart D. Johnson

Department of Mathematics, Cornell University, Ithaca, NY 14850, USA

Abstract. For the transformation $T: [0,1] \mapsto [0,1]$ defined by $T(x) = \lambda x(1-x)$ with $0 \leqq \lambda \leqq 4$, a λ is shown to exist for which T has no restrictive intervals, hence is sensitive to initial conditions, but for which no finite absolutely continuous invariant measure exists for T.

Introduction

A restrictive interval R for a one dimensional transformation T is one which eventually maps into itself; $T^n(R) \subset R$ for some n, the least such n being called the period of the interval. In the case where T is unimodal with negative Schwarzian derivative, John Guckenheimer showed that the absence of a restrictive interval implied an expansion property called sensitivity to initial conditions: $\exists \varepsilon \, \forall$ intervals $I \, \exists N$ such that $l(T^N(I)) > \varepsilon$. The same conclusion can be drawn in the case of a finite number of nested restrictive intervals by inducing on the smallest such interval. Thus a dichotomy exists for such maps T not having a period orbit; either T has sensitivity to initial conditions or there exists an infinite nested sequence of restrictive intervals creating an attracting invariant Cantor set upon which T is $1-1$. See Guckenheimer [5], or Collet and Eckmann [2], for a detailed exposition of these ideas.

Of interest is how the structure of invariant measures correspond to this dichotomy. Any invariant Cantor set will support an invariant measure, although it is not clear whether the Cantor sets in this context can have positive lebesgue measure. The question has been posed (Jakobson [6], Guckenheimer [5]) as to whether the absence of a restrictive interval is sufficient to conclude the existence of an absolutely continuous invariant measure. This is shown not to be the case in the following example.

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