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Kac-Moody Groups, Topology of the Dirac Determinant Bundle, and Fermionization

Jouko Mickelsson*

Research Institute for Theoretical Physics, University of Helsinki, SF-00260 Helsinki, Finland

Abstract. The relation between Kac-Moody groups and algebras and the determinant line bundle of the massless Dirac operator in two dimensions is clarified. Analogous objects are studied in four space-time dimensions and a generalization of Witten's fermionization mechanism is presented in terms of the topology of the Dirac determinant bundle.

1. Introduction

Loop groups and their central extensions in relation to Kac-Moody algebras have recently attracted great interest. The theory of loop groups is important among others in understanding completely integrable non-linear systems of KdV type [1], constructing solutions for self-dual Yang-Mills equations [2], string models in particle physics [3], and anomaly problems in quantum field theory [4].

The purpose of the present paper is twofold. First, we shall study the non-trivial $U(1)$ bundle P on the space ΩG of loops in a Lie group. This bundle has a Lie group structure such that the corresponding Lie algebra is a Kac-Moody algebra based on the Lie algebra of G. The structure of the group P has been earlier studied in detail in [1, 5, 6]. The physically important realization in [1] is that the sections of an associated line bundle E form in a natural way a Hilbert space which has a canonical realization as the Fock space for fermions in $1+1$ dimensions. There is a second construction of P directly in terms of local charts in ΩG, transition functions and local two-cocycles [6]. However, we feel that the subject is important enough to deserve a third construction; our construction is very simple and it makes clear the relation of sections of E to sections of the determinant line bundle of the $(1+1)$-dimensional Dirac operator.

The second main point in this paper is to clarify the topological and geometrical structure of the Dirac determinant line bundle in $3+1$ dimensions. Particular attention is paid to the case $G=SU(2)$; this case is important for

* Permanent address: Department of Mathematics, University of Jyväskylä, Seminaarinkatu 15, SF-40100 Jyväskylä, Finland