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Determinants of Laplacians*

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Abstract. The determinant of the Laplacian on spinor fields on a Riemann surface is evaluated in terms of the value of the Selberg zeta function at the middle of the critical strip. A key role in deriving this relation is played by the Barnes double gamma function.

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In recent years the problem of evaluating the determinants of certain elliptic operators has received considerable attention. The case of determinants of Laplacians on Riemann surfaces is crucial in Polyakov's string theory [P]. It also comes up in Ray-Singer's analytic torsion [R-S]. In recent papers D'Hoker and Phong [D-P 1, 2] and Kierlanczyk [KIE] computed these determinants of Laplacians in terms of special values of the Selberg zeta function. They succeeded except in the case of the Laplacian corresponding to the Dirac operator on spinors, for which they conjectured the determinant should be related to the Selberg zeta function at the middle of the critical strip. Our aim in this note is to prove that this is indeed so. The method followed by the above authors is to use the precise heat kernel for the hyperbolic plane, and the Selberg zeta function appears somewhat surprisingly. Our approach is more in the spirit of the analysis of the Selberg zeta function given in Hejhal [H], Selberg [S], and Vignéras [VI]. We show that the Selberg zeta function may be realized as a functional determinant (in fact as an analogue of a characteristic polynomial). The precise statement is given in Theorem 1. Our result follows from an asymptotic analysis in which a key role is played by Barnes' double gamma function and in particular Stirlings formula. The determinants in question are computed by specializing the parameter in Theorem 1.

Our notation will be as in D'Hoker and Phong [D-P 2]. We review it briefly. M is a compact Riemann surface of genus $h \geqq 2$ carrying a metric ds^2 of constant

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