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The Chiral Determinant and the Eta Invariant

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Abstract. For $\{\partial_y\}$, $y \in \mathbb{R}$, a one parameter family of invertible Weyl operators of possibly non-zero index acting on spinors over an even dimensional compact manifold X, we express the phase of the chiral determinant $\det \partial^\dagger_{-\infty}\partial_\infty$ in terms of the η invariant of a Dirac operator acting on spinors over $\mathbb{R} \times X$.

1. Statement of Results

Let X be a compact spin manifold of even dimension with spin bundle $S = S_+ \oplus S_- \to X$ and let $E \to X$ be a hermitian vector bundle over X. Let $\bar{S} = \mathbb{R} \times S$, $\bar{E} = \mathbb{R} \times S$ be the pullbacks of S and E to $\mathbb{R} \times X$ with the pull-back hermitian inner products, and let $\nabla^{\bar{E}}$ be a connection on $\bar{E}$. Thus $\nabla^{\bar{E}} = d_{\mathbb{R}} + \theta + \nabla^E_{(\cdot)}$, where $\theta \in \Omega^1(\mathbb{R}) \otimes C^\infty(X, \operatorname{End} E)$, and for each $u \in \mathbb{R}$, ∇^E_u is a connection on $E \to X$.

For $u \in \mathbb{R}$, let $\partial_u : C^\infty(X, S_+ \otimes E) \mapsto C^\infty(X, S_- \otimes E)$ and $D_u : C^\infty(X, S \otimes E) \mapsto C^\infty(X, S \otimes E)$ be the Weyl and Dirac operators coupled to the metric on X and the connection ∇^E_u on E. In the decomposition defined by $S = S_+ \oplus S_-$, $D_u = \begin{pmatrix} & \partial^\dagger_u \\ \partial_u & \end{pmatrix}$. Let H be the formally self-adjoint Dirac operator on $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ coupled to the connection $\nabla^{\bar{E}}$ on $\bar{E}$ and the product metric on $\mathbb{R} \times X$. Thus $H = i\Gamma\left(\frac{\partial}{\partial u} + \theta\left(\frac{\partial}{\partial u}\right)\right) + D_{(\cdot)}$, where Γ is the endomorphism of S with $\Gamma = \pm 1$ on $S_\pm$.

Assume

1. *For all* $u \in \mathbb{R}$, $\partial^\dagger_{-\infty}\partial_u$ *is invertible.*
2. *For* $|u|$ *large,* $\theta = 0$ *and* $d\nabla^E/du = 0$.

Condition 1 implies that for all u, $\operatorname{Ker} \partial_u = 0$ and $\operatorname{Ker} \partial^\dagger_u$ is a finite dimensional complement in L^2 of $\operatorname{Im} \partial_{-\infty}$. Condition 2 implies that for $|u|$ large, ∂_u is independent of u and H is invariant under translations in the $\mathbb{R}$ direction.

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