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Invariants for Smooth Conjugacy of Hyperbolic Dynamical Systems II

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Abstract. We show that the eigenvalues of the derivatives at periodic points form a complete set of invariants for smooth local conjugacy of Anosov diffeomorphisms of T^2.

0 Introduction

One of the most important results about structural stability is that if f, g are C^∞ Anosov diffeomorphisms of a compact manifold M which are sufficiently C^0 close (exactly how close depends on C^1 properties of both f, g) then there exists a homeomorphism h such that we have

$$f \circ h = h \circ g. \tag{1}$$

Moreover, the homeomorphism constructed in the theorem is C^0 close to the identity if f, g are C^0 close and is unique among those satisfying conditions of proximity to the identity.

It is a natural question to ask how smooth can h be.

It is known that h is C^α for some $\alpha > 0$. (This α is related to the contractive and expansive constants of f, g, so that the best α yielded by the proof is always smaller than 1.) C^1 conjugacy is harder. Indeed, if h were differentiable, and $f^n(x_0) = x_0$, we would have $g^n(h^{-1}(x_0)) = h^{-1}(x_0)$ and

$$Df^n(x_0) = Dh(h^{-1}(x_0))Dg^n(h^{-1}(x_0))Dh^{-1}(h^{-1}(x_0)) \quad \text{so that,} \tag{2}$$

$$\text{Spectrum } (Df^n(x_0)) = \text{Spectrum } (Dg^n(h^{-1}(x_0))) \text{ whenever } f^n(x_0) = x_0 \tag{3}$$

(A slightly more careful argument would show that (3) is also a necessary condition for h being Lipschitz.)

There are examples [An] that show that in general, conditions (3) are violated so that there is no hope of getting differentiable conjugacy without extra hypothesis, and indeed such examples played a major role in the proposal of [Sm] to restrict the study to continuous conjugacy.

However, there are very natural questions:

A) Suppose that (3) is met, is h differentiable?

B) Is the set of diffeomorphisms satisfying (3) a manifold?

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