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On Limit Theorems for the Bivariate (Magnetization, Energy) Variable at the Critical Point

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Abstract. The limiting (magnetization, energy) bivariate variable is studied for Ising ferromagnets at the critical point. The factorization property of the limiting bivariate moment generating function is shown to be intimately connected to critical point exponent inequalities and to the behaviour of the scaling limit near and at the critical point. The validity of this can be deduced from the study of the second and the fourth magnetization cumulants at zero external field. The limiting bivariate variable is exactly calculated at the critical point for the Curie–Weiss model (MF) and for the edge of a two-dimensional Ising ferromagnet wrapped on a cylinder. It is shown that the mean field case leads to a non-Gaussian limiting distribution in contradistinction with the particular Ising model we consider for which we obtain a product of two Gaussian probability distributions.

1. Introduction

Many recent investigations have been devoted to limit theorems for sums of dependent random variables occurring in classical spin models such as the magnetization and the energy variables [1–12]. The main difficulty encountered in this field has to be found in the non-independent character of the elementary spin variables which describe the models.

Let us indeed consider for definiteness a classical Ising ferromagnet on $\mathbb{Z}^d$ defined by the usual Hamiltonian

$$H(\{\sigma_\Lambda\}) = -\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i, \tag{1}$$

where $\Lambda \subset \mathbb{Z}^d, J_{ij} \geqq 0$ with $\sum_{j\in\mathbb{Z}^d} J_{ij} < +\infty$, $\{\sigma_\Lambda\}$ refers to a given configuration for Λ and h denotes the external field. The renormalized block magnetization variable $\tilde{M}_\Lambda$ and renormalized block energy variable $\tilde{E}_\Lambda$ are given by

$$M_\Lambda = \sum_{i\in\Lambda}\sigma_i, \tag{2.a}$$

$$\tilde{M}_\Lambda = (M_\Lambda - \langle M_\Lambda\rangle)/|\Lambda|^{\omega_1/2}, \tag{2.b}$$