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Quantum Toda Systems and Lax Pairs

N. Ganoulis

Physics Division, School of Technology, University of Thessaloniki, Greece

Abstract. We describe a general method for constructing a Lax pair representation of certain quantum mechanical systems that are integrable at the classical level. This is then used to find conserved quantities at the quantum level for the Toda systems.

1. Introduction

There is as fairly general method for constructing conserved quantities of classical mechanical systems in one and two dimensions (field-theories), that has been much studied and developed recently. In this method [1, 2] one starts with a Fundamental Poisson Bracket relation, that is, an expression for the Poisson bracket between the elements of a certain matrix, which is a function of the canonical variables of the dynamical system. One then finds, under certain conditions, that it is possible to get a family of conserved quantities, or Hamiltonians, in involution. A zero curvature condition for the "gauge potentials" (which are the auxiliary matrices constructed out of the canonical variables), plays an important role in this construction.

In the one dimensional case, this corresponds to showing that there is a Lax pair representation of the dynamical system, i.e. the classical equations of motion can be written in the form $\frac{dA}{dt}=[A, B]$, where A and B are matrices, functions of the canonical variables. The conserved quantites are then $\mathrm{Tr}\,A^N$ for any power N.

In this paper we will develop a similar approach for a quantum mechanical system. We consider the case when the Fundamental Poisson Bracket (FPB) goes over directly into a commutator bracket. We then show that there is a Lax pair representation of the quantum system.

This method has been developed with a particular application in mind. We wanted to construct conserved quantities for the quantum mechanical Toda system [3]. This is done by making use of the Lie algebraic properties of our "quantum mechanical" Lax pair. Our approach works uniformly for all the finite classical Lie algebras.

A first proof of integrability of the quantum Toda lattice was given by Kostant in [6]. Goodman and Wallach [7] give a recursive procedure for constructing integrals of the classical and quantum system. The reader is also referred to the review article [11], where several explicit formulae for first integrals can be found.