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Oscillations and Concentrations in Weak Solutions of the Incompressible Fluid Equations

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Abstract. The authors introduce a new concept of measure-valued solution for the 3-*D* incompressible Euler equations in order to incorporate the complex phenomena present in limits of approximate solutions of these equations. One application of the concepts developed here is the following important result: a sequence of Leray-Hopf weak solutions of the Navier-Stokes equations converges in the high Reynolds number limit to a measure-valued solution of 3-*D* Euler defined for all positive times. The authors present several explicit examples of solution sequences for 3-*D* incompressible Euler with uniformly bounded local kinetic energy which exhibit complex phenomena involving both persistence of oscillations and development of concentrations. An extensions of the concept of Young measure is developed to incorporate these complex phenomena in the measure-valued solutions constructed here.

Introduction

The Euler equations for an incompressible homogeneous fluid in n-space dimensions are given by

$$\frac{Dv}{Dt} = -\nabla p, \quad x \in R^n, \quad t > 0, \quad \operatorname{div} v = 0, \quad v(x,0) = v_0(x), \tag{1.1}$$

while the Navier-Stokes equations $\left(\text{with Reynolds number } \frac{1}{\varepsilon}\right)$ are given by

$$\frac{Dv^\varepsilon}{Dt} = -\nabla p^\varepsilon + \varepsilon \Delta v^\varepsilon, \quad x \in R^n, \quad t > 0, \quad \operatorname{div} v^\varepsilon = 0, \quad v^\varepsilon(x,0) = v_0(x). \tag{1.2}$$

Here $v = {}^t(v_1, \ldots, v_n)$ is the fluid velocity, $\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \cdot \nabla v$, is the convective derivative and p is the scalar pressure. The structure of solutions of the Euler

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