Commun. Math. Phys. 108, 489–526 (1987)

Communications in
Mathematical
Physics

Sharpness of the Phase Transition in Percolation Models*

Michael Aizenman** and David J. Barsky

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Abstract. The equality of two critical points – the percolation threshold p_H and the point p_T where the cluster size distribution ceases to decay exponentially – is proven for all translation invariant independent percolation models on homogeneous d-dimensional lattices ($d \geqq 1$). The analysis is based on a pair of new nonlinear partial differential inequalities for an order parameter $M(\beta, h)$, which for $h=0$ reduces to the percolation density P_∞ – at the bond density $p=1-e^{-\beta}$ in the single parameter case. These are: (1) $M \leqq h\partial M/\partial h + M^2 + \beta M \partial M/\partial\beta$, and (2) $\partial M/\partial\beta \leqq |J| M \partial M/\partial h$. Inequality (1) is intriguing in that its derivation provides yet another hint of a "φ^3 structure" in percolation models. Moreover, through the elimination of one of its derivatives, (1) yields a pair of ordinary differential inequalities which provide information on the critical exponents $\hat{\beta}$ and δ. One of these resembles an Ising model inequality of Fröhlich and Sokal and yields the mean field bound $\delta \geqq 2$, and the other implies the result of Chayes and Chayes that $\hat{\beta} \leqq 1$. An inequality identical to (2) is known for Ising models, where it provides the basis for Newman's universal relation $\hat{\beta}(\delta-1) \geqq 1$ and for certain extrapolation principles, which are now made applicable also to independent percolation. These results apply to both finite and long range models, with or without orientation, and extend to periodic and weakly inhomogeneous systems.

1. Introduction

There have traditionally been two different notions of a critical point in percolation models, corresponding to the boundaries of the low density and the high density regimes. For the standard one parameter percolation models on the d-dimensional square lattice $\mathbb{Z}^d$ (i.e. site percolation or nearest neighboor bond

---

* Research supported in part by the NSF Grant PHY-8605164

** Also in the Physics Department