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Details of the Non-Unitarity Proof for Highest Weight Representations of the Virasoro Algebra

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Abstract. We give an exposition of the details of the proof that all highest weight representations of the Virasoro algebra for $c<1$ which are not in the discrete series are non-unitary.

The Virasoro algebra is the infinite dimensional Lie algebra with generators L_n, $n\in\mathbb{Z}$, satisfying the commutation relations

$$[L_m, L_n]=(m-n)L_{m+n}+\tfrac{1}{12}c(m^3-m)\delta_{m+n,0}\,. \tag{1}$$

The number c is called the central charge. The Verma module $V(c,h)$ is the representation of the Virasoro algebra generated by a vector $|h\rangle$ satisfying

$$L_0|h\rangle=h|h\rangle\,,\qquad L_n|h\rangle=0\,,\qquad n>0\,, \tag{2}$$

and spanned by the linearly independent vectors $|h\rangle$ and

$$L_{-k_1}L_{-k_2}\dots L_{-k_n}|h\rangle\,,\qquad 1\leqq k_1\leqq k_2\leqq\dots\leqq k_n\,. \tag{3}$$

We assume that both c and h are real. In this case, a hermitian inner product on $V(c,h)$ is defined by $\langle h|h\rangle=1$, and $L_n^\dagger=L_{-n}$. Define, for p and q positive integers,

$$c(m)=1-\frac{6}{m(m+1)}\,,\qquad h_{p,q}(m)=\frac{((m+1)p-mq)^2-1}{4m(m+1)}\,. \tag{4}$$

The non-unitary theorem [1] is

Theorem 1. *For $c<1$ there are negative metric states in $V(c,h)$ if (c,h) does not belong to the discrete list*

$$c=c(m)\,,\qquad m=2,3,4\dots\,,\qquad h=h_{p,q}(m)\,,\qquad p+q\leqq m\,. \tag{5}$$

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