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Proof of Character-Valued Index Theorems*

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Abstract. A character-valued index is a generalization of the ordinary Dirac index to manifolds with nontrivial automorphism groups. A simple proof of the corresponding fixed-point theorem is presented which uses the techniques of supersymmetric quantum mechanics. This theorem relates the character-valued index to a topological integral of curvature forms on the fixed-point space of the automorphism in question.

1. Introduction

In recent years, topological methods have come to play an increasingly central role in theoretical physics, particularly in models with more than four spacetime dimensions. One of the most remarkable applications of topology to field theory is the Atiyah–Singer index theorem for the Dirac operator, [1] which describes the chiral content of the fermionic zero-modes in a particular background field configuration, containing both gravitational and gauge fields. This theorem contributed to an understanding of anomalies [2] and in particular how instantons can lead to the breaking of anomalous global symmetries [3]. Index theorems for families of field configurations [4] have improved our understanding of anomalous gauge symmetries.

The formal structure of topological indices is familiar to physicists in the form of Witten's supersymmetry index [5] $\mathrm{Tr}(-1)^F$. It measures the difference in the number of bosonic and fermionic states of zero energy, and, if it is not zero, indicates that supersymmetry cannot be spontaneously broken. Many authors [6] have noted the correspondence between the Witten index for supersymmetric quantum mechanics models and the Dirac index. This correspondence leads to a simple way of deriving the Atiyah–Singer index theorem and other related results.

Index theorems are particularly important in models in more than four spacetime dimensions. The spacetime then takes the form $M = M^4 \times K$, where M^4 is

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