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The Mumford Form and the Polyakov Measure in String Theory

A. A. Beilinson[1] and Yu. I. Manin[2]

[1] State Cardiology Center, Moscow, USSR
[2] Steklov Mathematical Institute, Moscow, USSR

Abstract. An explicit formula is derived for the Mumford form on the moduli space of algebraic curves. According to the Belavin-Knizhnik theorem, this gives a formula for the Polyakov bosonic string measure.

Introduction

1. Polyakov's String

The quantum bosonic Polyakov string theory is defined by a path integral taken over random surfaces in the (Euclidean) d-dimensional space $\mathbb{R}^d$. The partition function of the closed string has a perturbation series expansion $Z = \sum_{g \geqq 0} Z_g$:

$$\begin{aligned} Z_g &= e^{\beta(2-2g)} \int e^{-J(x,\gamma)} Dx\, D\gamma\,, \\ J(x,\gamma) &= \int_N d^2 z \sqrt{|\gamma|}\, \gamma^{ab} \partial_a x^\mu \partial_b x^\mu\,. \end{aligned} \tag{1}$$

Here N is a fixed compact oriented surface of genus g (="loop number"), z^a are local coordinates on it, $x=(x^\mu)$ is a map from N to $\mathbb{R}^d$, $\gamma_{ab} dz^a dz^b$ is a metric on N.

On the space of classical fields (x,γ) a gauge group $C \ltimes D$ acts, leaving the classical action $J(x,\gamma)$ invariant. It is a semidirect product of the diffeomorphism group D of N and of the conformal group C (=real-valued positive functions on N). Using this action, we can reduce (1) to a finite-dimensional integral in the following way. First, the integral over x's is Gaussian, hence it equals $(\det' \Delta_{0\gamma})^{-d/2}$, where $\Delta_{0\gamma}$ is the Laplace operator on the functions on N, corresponding to γ, $\det'$ its determinant without zero modes, regularized, say, by the formula $\det' \Delta_{0\gamma} = \exp(-\zeta'_\Delta(0))$. The remaining integration over the space $\operatorname{Met} N$ of γ's then reduces, via the Faddeev-Popov trick, to an integral over $\operatorname{Met} N / C \ltimes D$, which is the same as the moduli space of Riemannian surfaces (or complex algebraic curves) of genus g. As is well known, this moduli space M_g is a complex variety of complex dimension 0 for $g=0$, 1 for $g=1$, $3g-3$ for $g \geqq 2$.