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Symbolic Dynamics for the Renormalization Map of a Quasiperiodic Schrödinger Equation

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Abstract. A rigorous analysis is given of the dynamics of the renormalization map associated to a discrete Schrödinger operator H on $l^2(\mathbb{Z})$, defined by $H\psi(n)=\psi(n+1)+\psi(n-1)+Vf(n\sigma)\psi(n)$, where V is a real parameter, f is a certain discontinuous period-1 function, and $\sigma=(-1+\sqrt{5})/2$ is the golden mean. The renormalization map for H is a diffeomorphism, T, of $\mathbb{R}^3$, preserving a cubic surface S_V. For $V\geqq 8$ we prove that the non-wandering set of the restriction of T to S_V is a hyperbolic set, on which T is conjugate to a subshift on six symbols. It follows from results in dynamical systems theory that the optimally approximating periodic operators to H have spectra which obey a global scaling law. We also define a set which we call the pseudospectrum" of the operator H. We prove it to be a Cantor set of measure zero, and obtain bounds on its Hausdorff dimension. It is an open question whether the pseudospectrum coincides with the spectrum of H.

Introduction

There has been much interest in Schrödinger operators with a quasiperiodic potential (see [18, 19, 26–28, 33] and references therein). These operators have numerous physical applications. For example, they describe the electron spectrum of periodic crystals in a magnetic field [15], and the electron and phonon spectrum of the recently discovered quasi-crystals [3]. They also arise in the linear stability of motions in classical mechanics [1]. Operators with quasi periodic potential also pose very interesting questions for the functional analyst [33]. They are in some sense intermediate between operators with periodic potential and operators with random potential. Periodic potentials are well known to lead to absolutely continuous "band spectra" and extended eigenstates [31], whereas random potentials lead to pure point spectra and localized eigenstates, in one dimension [20]. In the quasiperiodic case the general belief is that the spectra are Cantor sets. At present, the only theorems in this direction are for a generic set of potentials, which are very well approximated by periodic potentials [2]. In this case, the