Commun. Math. Phys. 107, 213–231

Communications in Mathematical Physics

On the Existence of Fixed Points of the Composition Operator for Circle Maps

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Abstract. In the theory of circle maps with golden ratio rotation number formulated by Feigenbaum, Kadanoff, and Shenker [FKS], and by Ostlund, Rand, Sethna, and Siggia [ORSS], a central role is played by fixed points of a certain composition operator in map space. We define a common setting for the problem of proving the existence of these fixed points and of those occurring in the theory of maps of the interval. We give a proof of the existence of the fixed points for a wide range of the parameters on which they depend.

1. Introduction

Fixed points for composition operators are now known to exist in a multitude of situations: maps of the interval, dissipative maps of $\mathbb{R}^n$, area preserving maps of the plane and circle maps all possess fixed points. In this paper, we try to connect the cases of interval maps and circle maps (with golden rotation number) by giving an interpolation between the two. In fact, there is a two parameter family of problems, namely to find solutions of the equations

$$\phi(x) = -\frac{1}{\lambda}\phi\left(\frac{1}{\lambda^{\nu-1}}\phi(\lambda^\nu x)\right) \tag{1.1}$$

with

$$\phi(x) = f(x^r), \qquad \phi\left(\frac{1}{\lambda^{\nu-1}}\right) = -\lambda, \qquad \phi(0) = 1, \qquad \lambda \in (0,1),$$

and f analytic on $[0,1]$. Here, the two parameters are r and ν. The case $\nu = 1$ corresponds to maps of the interval, while the case $\nu = 2$ corresponds to circle maps. The value $r = 2$ is of main interest for the physics corresponding to the case $\nu = 1$, and $r = 3$ is of interest when $\nu = 2$. For the possible occurrence of different r in dynamical systems, see [ACT]. [A more precise statement of the problem is embodied in Eqs. (1.19), (1.20) below.]

Consider the region D given by

$$D = \{(\lambda, \nu) \in \mathbb{R}^2 : 1 \leqq \nu \leqq 2, 0 < \lambda < 1, \lambda^\nu + \lambda^{\nu-1} - 1 > 0\}.$$