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Vertex Operators for Non-Simply-Laced Algebras

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Abstract. A vertex operator construction is given for the level one representations of the affine Kac-Moody algebras associated with non-simply-laced finite-dimensional Lie algebras, using free boson and interacting fermion fields. The fermion fields are constructed explicitly and a detailed discussion is given of the theory of the cocycles necessary for this and other vertex operator constructions. The construction is related in detail to the folding of Dynkin diagrams and a generalisation of it yields Freudenthal's magic square.

1. Introduction

Since Frenkel and Kac [1] and, independently, Segal [2] constructed level one represenations of untwisted affine Kac-Moody algebras $\hat{g}$, associated with simple Lie algebras g which are simply-laced, using the vertex operators of string theory, a corresponding construction for the non-simply-laced case has been sought. The cases where g is simply laced, i.e. all roots have a common length, are $g = \mathrm{su}(r+1)$, $\mathrm{so}(2r)$, E_6, E_7, and E_8, and the non-simply laced cases are $g = \mathrm{so}(2r+1)$, $\mathrm{sp}(r)$, each if $r \geqq 2$, G_2 and F_4. (For a review of terminology and results on Kac-Moody and Virasoro algebras, developed in relation to their applications in quantum physics, see [3].) In this paper we provide a vertex operator construction for the non-simply laced case.

There are a number of reasons for wanting such a construction. For example, it gives us a more uniform approach to the most basic representations of all the untwisted algebras $\hat{g}$. It involves the introduction of fermion operators associated with the short roots. The Frenkel-Kac-Segal (FKS) construction enables one to build non-abelian internal symmetries into string theories in an intrinsic way by compactifying some of the dimensions in which the string moves to form the maximal torus of a simply-laced group [4]. This has been exploited to construct potentially realistic unified string theories of particle interactions [5]. These theories involve fermions and the present generalisation of the FKS construction by the

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