Inequalities for the Schatten $p$-Norm. IV

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Abstract. We prove some inequalities for the Schatten $p$-norm of operators on a Hilbert space. It is shown, among other things, that if $A$, $B$, and $X$ are operators such that $A + B \geq |X|$ and $A + B \geq |X^*|$, then $\|AX + XB\|_p^p + \|AX^* + X^*B\|_p^p \geq 2\|X\|_p^{2p}$ for $1 \leq p < \infty$, and $\max(\|AX + XB\|, \|AX^* + X^*B\|) \geq \|X\|^2$.

Also, for any three operators $A$, $B$, and $X$,

$$\|AX + XB\|_p^p + \|AX^* + X^*B\|_p^p \leq \|AX - XB\|_p^p + \|A^*X - X^*B\|_p^p,$$

1. Introduction

In their work on free states of the canonical anticommutation relations, Powers and Størmer [9, Lemma 4.1] proved that if $A$ and $B$ are positive operators on a Hilbert space $H$, then $\|A^{1/2} - B^{1/2}\|_2^2 \leq \|A - B\|_1$. Also, in studying the quasi-equivalence of quasifree states of canonical commutation relations, Araki and Yamagami [2, Theorem 1] proved that if $A$ and $B$ are operators on a Hilbert space $H$, then $\|A - B\|_1 \leq 21/2 \|A - B\|_2$. This has been recently generalized so that $\|A - B\|_2 + \|A - B\|_2^2 \leq 2 \|A - B\|_2^2$ [7, Theorem 2].

The purpose of this paper, which is in the same spirit as those of [5–7], is to extend these inequalities to commutator versions and to show that in some cases the trace norm can be replaced by a general $p$-norm. In particular it will be shown that for positive operators $A$ and $B$, $\|A^{1/2} - B^{1/2}\|_p^2 \leq \|A - B\|_p$ for $1 \leq p \leq \infty$.

Let $H$ be a separable complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on $H$. Let $K(H)$ denote the closed two-sided ideal of compact operators on $H$. For any compact operator $A$, let $s_1(A), s_2(A), \ldots$ be the eigenvalues of $|A| = (A^*A)^{1/2}$ in decreasing order and repeated according to multiplicity. A compact operator $A$ is said to be in the Schatten $p$-class $C_p$ ($1 \leq p < \infty$), if $\sum s_i(A)^p < \infty$. The Schatten $p$-norm of $A$ is defined by $\|A\|_p = (\sum s_i(A)^p)^{1/p}$. This norm makes $C_p$ into a Banach space. Hence $C_1$ is the trace class and $C_2$ is the Hilbert–Schmidt class. It is reasonable to let $C_\infty$ denote the ideal of compact operators $K(H)$, and $\|\cdot\|_\infty$ stand for the usual operator norm.

If $A \in C_p$ ($1 \leq p < \infty$) and $\{e_i\}$ is any orthonormal set in $H$, then $\|A\|_p^p \geq \sum |\langle Ae_i, e_i \rangle|^p$. More generally, if $\{E_i\}$ is a family of orthogonal projections satisfying $E_i E_j = \delta_{ij} E_i$, then $\|A\|_p^p \geq \sum \|E_i A E_i\|_p^p = \|\sum E_i A E_i\|_p^p$, and for $p > 1$ equality will hold if and only if $A = \sum E_i A E_i$. Moreover, if $\sum E_i = 1$ (the identity operator) and $p = 2$, then $\|A\|_2^2 = \sum \|E_i A E_i\|_2^2$. One more fact that will be needed in