Commun. Math. Phys. 106, 569–580 (1986)

Communications in Mathematical Physics

The Nonlinear Schrödinger Limit of the Zakharov Equations Governing Langmuir Turbulence

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Abstract. We consider the initial value problem for the Zakharov equations

$$(\mathrm{Z}) \qquad \begin{aligned} &\frac{1}{\lambda^2} n_{tt} - \Delta(n+|E|^2) = 0 \qquad && n(x,0) = n_0(x) \\ & && n_t(x,0) = n_1(x) \\ &iE_t + \Delta E - nE = 0 && E(x,0) = E_0(x) \end{aligned}$$

($x \in \mathbb{R}^k, k = 2, 3, t \geqq 0$) which model the propagation of Langmuir waves in plasmas. For suitable initial data solutions are shown to exist for a time interval independent of λ, a parameter proportional to the ion acoustic speed. For such data, solutions of (Z) converge as $\lambda \to \infty$ to a solution of the cubic nonlinear Schrödinger equation

$$(\mathrm{CSE}) \qquad iE_t + \Delta E + |E|^2 E = 0.$$

We consider both weak and strong solutions. For the case of strong solutions the results are analogous to previous results on the incompressible limit of compressible fluids.

I. Introduction

The Zakharov equations [Z, GTWT],

$$\frac{1}{\lambda^2} n_{tt} - \Delta(n + |E|^2) = 0, \tag{1.1}$$

$$iE_t + \Delta E - nE = 0, \tag{1.2}$$

$E: \mathbb{R}_x^k \times \mathbb{R}_t^+ \to \mathbb{C}^k, n: \mathbb{R}_x^k \times \mathbb{R}_t^+ \to \mathbb{R}$, describe the propagation of Langmuir waves in plasmas. The complex vector E denotes the slowly varying envelope of the highly oscillatory electric field, and n is the fluctuation in the ion-density about its equilibrium value. The parameter λ is proportional to the ion acoustic speed. Other physical parameters have been removed by scaling.

Formally letting λ tend to infinity in (1.1) yields the equation $\Delta(n + |E|^2) = 0$, which implies $n = -|E|^2$ if n and $|E|^2$ are square-integrable. Substitution of this

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