Commun. Math. Phys. 106, 353–381 (1986)

Communications in
Mathematical
Physics

Bifurcations of Circle Maps: Arnol'd Tongues, Bistability and Rotation Intervals

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Abstract. We study the bifurcations of two parameter families of circle maps that are similar to $f_{b,w}(x)=x+w+(b/2\pi)\sin(2\pi x)\,(\mathrm{mod}\,1)$. The bifurcation diagram is constructed in terms of sets T_r, where T_r is the set of parameter values (b, w) for which $f_{b,w}$ has an orbit with rotation number r. We show that the known structure when $b<1$ (for r rational, T_r is an Arnol'd tongue and for r irrational, it is an arc) extends nicely into the region $b>1$, where $f_{b,w}$ is no longer injective and can have an interval of rotation numbers. Specifically, the tongues overlap in a uniform, monotonic manner and for r irrational, T_r opens into a tongue. Our other theorems give information about the dynamics of $f_{b,w}$ (e.g., bistability or aperiodicity) for (b, w) in various subsets of parameter space.

0. Introduction

In this paper we construct the bifurcation diagram of two parameter families of circle maps which are similar to what has been termed the "canonical" family [16], $f_{b,w}(x)=x+w+(b/2\pi)\sin(2\pi x)\,(\mathrm{mod}\,1)$. These families are used to model various periodically forced nonlinear oscillators (e.g. [8, 17, 23]) and in the renormalization group analysis of the transition from quasiperiodicity to chaos [9, 22, 24]). When $b<1$, $f_{b,w}$ is a C^∞ circle diffeomorphism. The dynamics of such maps are well understood from the classic work of Poincaré and Denjoy. All orbits rotate at the same asymptotic rate, so one may define the rotation number of the diffeomorphism which essentially classifies the dynamics. When $b>1$ however, $f_{b,w}$ is no longer injective and can display the dynamic complexity which is well known for maps of the interval. Different orbits can rotate at different asymptotic rates, and $f_{b,w}$ can have an interval of rotation numbers [18].

Of particular interest in the bifurcation theory of dynamical systems is the description of the transition from simple to complicated or "chaotic" dynamics. In studying this transition in the canonical family it is natural to examine the

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