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On a Regularity Problem Occurring in Connection with Anosov Diffeomorphisms

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Abstract. Let $\mathcal{M}$ be a C^∞-manifold and $\mathcal{F}_s$ and $\mathcal{F}_u$ be two Hölder foliations, transverse, and with uniformly C^∞ leaves. If a function f is uniformly C^∞ along the leaves of the two foliations, then it is C^∞ on $\mathcal{M}$. The proof is elementary.

It is well known that a function whose restrictions to lines parallel to the coordinate axes are uniformly $\mathscr{C}^\infty$ is itself a $\mathscr{C}^\infty$-function. The problem amounts to reconstructing mixed derivatives and can be very easily done using Fourier transform.

In studying Anosov diffeomorphisms a similar sort of problem arises. In [LMM] it was observed that, very frequently, one could prove existence of derivatives along the stable and unstable manifolds which are smooth. However, the corresponding foliations are only Hölder, and it is not clear how to reduce to the original case as when the foliations are $\mathscr{C}^\infty$. In [LMM] however, the authors succeeded in giving a proof of the global smoothness of their functions by using elliptic theory, as well as another regularity property of their foliations, which is not easy to check [LMM]. We shall present an alternate approach to these sort of results. It won't require the extra regularity condition on the foliations, and therefore will lend itself to other kinds of applications.

We come to the statement of our theorem, which is conjectured in [LMM], at least implicitly.

Let $\mathcal{M}$ be a $\mathscr{C}^\infty$-manifold. We shall denote by $\mathcal{F}$ a foliation of $\mathcal{M}$ and for each $M \in \mathcal{M}$, $\mathcal{L}^M$ will be the leaf of $\mathcal{F}$ containing M. We assume that the leaves are uniformly $\mathscr{C}^\infty$ and that T^M, the tangent space at M of $\mathcal{L}^M$, is a Hölder function of M. We now suppose that we have two such foliations which are transverse.

Theorem. *If $f: \mathcal{M} \to \mathbb{R}$ is $\mathscr{C}^{n,\alpha}$ along the leaves of $\mathcal{F}_s$ and $\mathcal{F}_u$ for some $\alpha > 0$ uniformly, then f is $\mathscr{C}^{n,\beta}$ for some $\beta > 0$.*

Corollary. *If $f: \mathcal{M} \to \mathbb{R}$ is $\mathscr{C}^\infty$ along the leaves of $\mathcal{F}_s$ and $\mathcal{F}_u$ uniformly, then f is $\mathscr{C}^\infty$.*

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