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The Analysis of Elliptic Families. I. Metrics and Connections on Determinant Bundles

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Abstract. In this paper, we construct the Quillen metric on the determinant bundle associated with a family of elliptic first order differential operators. We also introduce a unitary connection on λ and calculate its curvature. Our results will be applied to the case of Dirac operators in a forthcoming paper.

In [Q2], Quillen gave a construction of a metric and of a holomorphic connection on the determinant bundle of a family of $\bar{\partial}$ operators. On the other hand, Bismut gave in [B1] a heat equation proof of the Atiyah–Singer Index Theorem for families of Dirac operators [AS1] using the superconnection formalism of Quillen [Q1]. In this paper, we extend the construction of Quillen [Q2] to the case of an arbitrary family of first order elliptic differential operators.

More precisely, let $M \xrightarrow{Z} B$ be a compact fibering of manifolds and let D_+ be a family of first order elliptic differential operators. D_+ can be considered as a smooth section of $\mathrm{Hom}(H_+^\infty, H_-^\infty)$, where H_+^∞, H_-^∞ are infinite dimensional Hermitian bundles over B. If λ is the line bundle $(\det \operatorname{Ker} D_+)^* \otimes (\det \operatorname{Coker} D_+)$, we construct a metric and a unitary connection on λ, and we calculate the corresponding curvature.

To explain the construction, let us temporarily assume that H_+^∞, H_-^∞ are instead finite dimensional Hermitian bundles over B which have the same dimension. In this case λ can be identified with $(\det H_+^\infty)^* \otimes \det H_-^\infty$, and so is naturally endowed with a Hermitian metric $\|\cdot\|$. Clearly $\det D_+$ is a section of λ.

Let D_- be the adjoint of D_+, and set

$$H^\infty = H_+^\infty \oplus H_-^\infty; \quad D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}. \tag{0.1}$$

Then

$$\|\det D_+\| = [\det D_- D_+]^{1/2} = [\det D^2]^{1/4}. \tag{0.2}$$

Also if H_+^∞, H_-^∞ are endowed with a unitary connection $\tilde{\nabla}^u$, λ is also endowed with a unitary connection ${}^1\nabla$. Where D_+ is invertible, we have for $Y \in TB$,

$$\tilde{\nabla}_Y^u \det D_+ = \det D_+ \operatorname{Tr}[D_+^{-1} \tilde{\nabla}_Y^u D_+]. \tag{0.3}$$

By [Q1], the graded algebra $\operatorname{End} H^\infty$ is endowed with a trace Tr and a supertrace