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Markov Quantum Semigroups Admit Covariant Markov C^*-Dilations

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Abstract. Through a Daniell-Kolmogorov type construction, to any Markov quantum semigroup on a C^*-algebra there is associated a quantum stochastic process which is a dilation of the semigroup, and satisfies a covariance rule which implies the weak Markov property.

Introduction

In the classical framework, a Markov semigroup is a semigroup $(P_t)_{t \geqq 0}$ of probability transitions on a (n eventually compact) space X. The Daniell-Kolmogorov construction (cf. [3, Sect. I.2]) is a natural procedure for associating to such a semigroup a strong Markov process which dilates it.

The simplest way of viewing this construction is to build a family $(\mu^x)_{x \in X}$ of probability measures on the space $X^{\mathbb{R}_+}$ of all (borel) trajectories as an inductive limit of the measures $\mu^x_{t_1, \dots, t_n}$ on $X^{\{t_1, \dots, t_n\}}$ $(t_1 < \dots < t_n)$ defined by

$$\mu^x_{t_1, \dots, t_n}(f) = \int_X f(x_1, \dots, x_n) P_{t_n - t_{n-1}}(x_{n-1}, dx_n) \dots P_{t_2 - t_1}(x_1, dx_2) P_{t_1}(x, dx).$$

More algebraically, consider the P_t as positive maps from $C(X)$ (the algebra of continuous functions on X) into itself, assuming they preserve the class of continuous functions. For any t choose a C^*-algebra A_t isomorphic to $C(X)$; then, for $t_1 < t_2$, consider the conditional expectation ε_{t_2, t_1} from $A_{t_1} \otimes A_{t_2}$ onto A_{t_1} characterized by

$$\varepsilon_{t_2, t_1}(f_2 \otimes f_1) = P_{t_2 - t_1}(f_2) f_1 . \tag{0.1}$$

On the tensor product $A_{t_n} \otimes \dots \otimes A_{t_1}$, one defines a family $(E_{t_k})_{k=1, \dots, n}$ of conditional expectations onto the sub C^*-algebra $A_{t_k} \otimes \dots \otimes A_{t_1}$ through the induction formula

$$E_{t_k} = (\varepsilon_{t_{k+1}, t_k} \otimes \text{identity}) \circ E_{t_{k+1}} .$$

The Markov process lies in the inductive limit of the C^*-algebras $A_{t_n} \otimes \dots \otimes A_{t_1}$ along the filter of finite subsets $\{t_1, \dots, t_n\}$ of $\mathbb{R}_+$. Its filtration is given by the inductive limit of the $E_t (t \in (t_1, \dots, t_n\})$, and a time evolution $(\sigma_s)_{s \geqq 0}$ is provided by