Commun. Math. Phys. 106, 1–40 (1986)

Communications in Mathematical Physics

Theta Functions, Modular Invariance, and Strings

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Abstract. We use Quillen's theorem and algebraic geometry to investigate the modular transformation properties of some quantities of interest in string theory. In particular, we show that the spin structure dependence of the chiral Dirac determinant on a Riemann surface is given by Riemann's theta function. We use this result to investigate the modular invariance of multiloop heterotic string amplitudes.

1. Introduction

Two-dimensional quantum field theories have served as toy models in attempts to understand more complicated four-dimensional theories. The two dimensional theories capture many essential features of higher dimensions, without sharing the complexities of higher dimensions. Certain features of $2d$ QFT's such as Bose-Fermi equivalence have led to a large number of exactly solvable theories.

In string theories, two-dimensional conformal QFT plays an even more important role [1–3]. The string sweeps out a surface as it moves through space-time, and therefore, the first quantized theory corresponds to a two dimensional QFT. The string can sweep out a surface with any number of handles. Whereas in most well known results one considers the underlying space to be R^2 (or $R \times S^1$, or $S^1 \times S^1$ corresponding to periodic boundary conditions in space or time), for string theories one must consider the space to be an arbitrary Riemann surface. Thus, understanding multiloop string amplitudes requires an understanding of QFT on a Riemann surface. In the general case, few explicit facts are known.

Many questions remain unanswered in string theory. For instance, how can we prove the vanishing of the cosmological constant in superstring theories? This has been shown explicitly at 1-loop. For higher loops, even though there is an indirect argument for the vanishing of the cosmological constant [3, 4], one would like to show this important fact more directly.

There are other issues: What happens to the string amplitude in the limit of a degenerating Riemann surface? Are superstring theories finite? Is the perturbation