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Existence and Partial Regularity of Static Liquid Crystal Configurations

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Abstract. We establish the existence and partial regularity for solutions of some boundary-value problems for the static theory of liquid crystals. Some related problems involving magnetic or electric fields are also discussed.

Introduction

The equilibrium configuration of a liquid crystal may be described in terms of its optical axis, a unit vector field n defined on the region Ω in $\mathbb{R}^3$ occupied by the material (see [E]). For a nematic liquid crystal, the Oseen–Frank free energy density W is given by

$$2W(\nabla n, n) = \kappa_1(\operatorname{div} n)^2 + \kappa_2(n \cdot \operatorname{curl} n)^2 + \kappa_3|n \times \operatorname{curl} n|^2 + (\kappa_2 + \kappa_4)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2], \tag{0.1}$$

where the constants κ_1, κ_2, κ_3, and κ_4 are generally assumed to satisfy

$$\kappa_1 > 0, \quad \kappa_2 > 0, \quad \kappa_3 > 0, \quad \kappa_2 \geqq |\kappa_4|, \quad \text{and} \quad 2\kappa_1 \geqq \kappa_2 + \kappa_4.$$

(Here we will assume only that κ_1, κ_2, and κ_3 are positive.)

The principal questions we shall discuss are the existence and partial regularity of a vectorfield n with the property that

$$\mathscr{W}(n) = \inf \mathscr{W}(u), \quad \text{where} \quad \mathscr{W}(u) = \int_\Omega W(\nabla u, u)dx, \tag{0.2}$$

and where the infimum is taken over all $u: \Omega \to \mathbb{S}^2$ having prescribed boundary values n_0 on $\partial\Omega$.

The existence of a minimizer $n \in H^1(\Omega, \mathbb{S}^2)$ by direct methods is presented in Sect. 1. The first ingredient of the proof is to establish that the class of competing functions is nonempty. The second involves certain coerciveness estimates for the functional $\mathscr{W}$. Of relevance here is the observation by C. Oseen, and later independently by J. L. Ericksen, that the last term in $\mathscr{W}$ is a surface energy in the

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