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Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories

II. Scattering States

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Abstract. The low-lying energy values associated to energy eigenstates describing two stable particles enclosed in a (space-like) box of size L are shown to be expandable in an asymptotic power series of $1/L$. The coefficients in these expansions are related to the appropriate elastic scattering amplitude in a simple and apparently universal manner. At low energies, the scattering amplitude can thus be determined, if an accurate calculation of two-particle energy values is possible (by numerical simulation, for example).

1. Introduction

This paper is a continuation of [1], where I have determined the size dependence of the stable particle masses in quantum field theories enclosed in an $L \times L \times L$ box with periodic boundary conditions. The objective here is, to find out how the energy eigenstates describing two (unbound) stable particles behave in finite volume and in particular how the associated energy values vary with L. The motivation for this investigation is at least two-fold. First, in numerical simulations of lattice theories, it is helpful to have some a priori knowledge about the distribution of the low-lying energy values to perform the spectral analysis of correlation functions and to correctly interpret the energy spectrum so determined. Secondly, the formulae established in this paper relate the size dependence of the two-particle energies to the corresponding elastic scattering amplitudes and thus make the latter accessible for calculational schemes, which need a finite volume for technical reasons and which are hence unable to deals with scattering processes directly. To compute low-energy scattering amplitudes via the energy spectrum in finite volume appears to be a rather complicated way to proceed, but in the context of numerical simulations of lattice gauge theories, for example, no other practical method is presently available.

In finite volume, the particle momenta are quantized and the spectrum of energies of two-particle states with zero total momentum is therefore discrete. As $L \to \infty$, the spacing between these levels goes to zero and their density grows proportionally to the volume. An important point to note is that the level spacing