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Sufficient Subalgebras and the Relative Entropy of States of a von Neumann Algebra

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Abstract. A subalgebra M_0 of a von Neumann algebra M is called weakly sufficient with respect to a pair (ϕ, ω) of states if the relative entropy of ϕ and ω coincides with the relative entropy of their restrictions to M_0. The main result says that M_0 is weakly sufficient for (ϕ, ω) if and only if M_0 contains the Radon-Nikodym cocycle $[D\phi, D\omega]_t$. Other conditions are formulated in terms of generalized conditional expectations and the relative Hamiltonian.

Introduction

Suppose that an experiment is described by a measurable space $(X, \mathscr{S})$ and the outcome of the experiment is governed by a probability measure μ on $\mathscr{S}$. In mathematical statistics the probability distribution μ belongs to a family Θ of probability distributions. A sub-σ-algebra $\mathscr{R}$ of $\mathscr{S}$ can be considered as application of a statistics or an indirect observation. $\mathscr{R}$ is defined to be sufficient with respect to the family Θ if the conditional distribution of $\mu \in \Theta$ does not depend on μ. More precisely, for every $S \in \mathscr{S}$ there exist an $\mathscr{R}$-measurable function ξ_S such that $\int_R \xi_S \, d\mu = \mu(R \cap S)$ for every $R \in \mathscr{R}$ and $\mu \in \Theta$ (see [12 or 7, 28]). In mathematical statistics the case $|\Theta| = 2$ is called the discrimination between two statistical hypotheses.

Let $\Theta = \{\mu, \nu\}$, and assume that $\mu, \nu \ll \lambda$. Halmos and Savage ([12]) proved that $\mathscr{R}$ is sufficient for $\{\mu, \nu\}$ if and only if the function

$$\frac{d\mu}{d\lambda} \Big/ \frac{d\nu}{d\lambda}$$

is $\mathscr{R}$-measurable. Another equivalent formulation due to Kullback and Leibler [18] is based on the relative entropy of measures. Namely, $\mathscr{R}$ is sufficient if and only if

$$S(\mu, \nu) = S(\mu_0, \nu_0),$$

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