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The Becker-Döring Cluster Equations: Basic Properties and Asymptotic Behaviour of Solutions

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Abstract. Existence and uniqueness results are established for solutions to the Becker-Döring cluster equations. The density ϱ is shown to be a conserved quantity. Under hypotheses applying to a model of a quenched binary alloy the asymptotic behaviour of solutions with rapidly decaying initial data is determined. Denoting the set of equilibrium solutions by $c^{(\varrho)}$, $0 \leqq \varrho \leqq \varrho_s$, the principal result is that if the initial density $\varrho_0 \leqq \varrho_s$ then the solution converges strongly to $c^{(\varrho_0)}$, while if $\varrho_0 > \varrho_s$ the solution converges weak* to $c^{(\varrho_s)}$. In the latter case the excess density $\varrho_0 - \varrho_s$ corresponds to the formation of larger and larger clusters, i.e. condensation. The main tools for studying the asymptotic behaviour are the use of a Lyapunov function with desirable continuity properties, obtained from a known Lyapunov function by the addition of a special multiple of the density, and a maximum principle for solutions.

1. Introduction

Consider a system of a large number of clusters of particles that can coagulate to form larger clusters or fragment to form smaller ones. Becker and Döring (1935) proposed an infinite system of ordinary differential equations as a model for the time evolution of the distribution of cluster sizes for such a system. In its original form this system treated the number of one-particle clusters as fixed: it did not take into account the depletion of the number of one-particle clusters as larger clusters are formed. A modified version of these equations allowing for depletion, which we still refer to as the Becker-Döring equations, was described by Penrose and Lebowitz (1979). In this paper we make a rigorous study of some fundamental properties of solutions to the (modified) Becker-Döring equations, and in particular analyze aspects of the asymptotic behaviour of solutions as time $t \to \infty$.

If $c_r(t) \geqq 0$, $r = 1, 2, \ldots$, denotes the expected number of r-particle clusters per unit volume at time t, then the Becker-Döring equations can be written in the form

$$\begin{aligned} \dot{c}_r &= J_{r-1}(c) - J_r(c), \qquad r \geqq 2, \\ \dot{c}_1 &= -J_1(c) - \sum_{r=1}^{\infty} J_r(c), \end{aligned} \tag{1.1}$$