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Map Dependence of the Fractal Dimension Deduced from Iterations of Circle Maps

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Abstract. Every orientation preserving circle map g with inflection points, including the maps proposed to describe the transition to chaos in phase-locking systems, gives occasion for a canonical fractal dimension D, namely that of the associated set of μ for which $f_\mu = \mu + g$ has irrational rotation number. We discuss how this dimension depends on the order r of the inflection points. In particular, in the smooth case we find numerically that $D(r) = D(r^{-1}) = r^{-1/8}$.

1. Introduction

Mathematical models for periodically stimulated oscillators are usually formulated as a system of coupled differential equations [1–5]. The associated Poincaré map gives the oscillator state at time n/v as a function of the state at time $(n-1)/v$, where v is the external frequency. In appropriate limits it has often been possible to reduce this map to a one-dimensional map of the form of those we consider here [2], [6–8].

The investigation of these circle maps has been particularly useful in studying the transition to chaos [9–11]. The fractal obtained along the critical line defined by the points where chaos sets in, is described by the fractal dimension [12] obtained from iterations of a circle map [5], and this dimension seems to be universal [9, 10]. As an example, the transition to hysteresis and chaos of the resistively shunted Josephson junction modulated by an rf microwave signal [13] can be modelled by the behavior of a circle map which passes from invertibility to non-invertibility through development of an inflection point of order three [8, 14]. This transition gives occasion for a complete devil's staircase structure [12], where the fractal dimension of the associated Cantor set is $D = 0.87$ [10].

In this paper we study numerically maps with inflection points with orders other than three. In particular we find that the related fractal dimension varies like the $1/8^{\text{th}}$ power of the order. In Sect. 2 we define a set G of circle maps, and in Sect. 3 we report the results of a numerical investigation of the fractal dimension of the