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The Effective Potential as an Energy Density: The One Phase Region

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Abstract. An explicit formula is given relating the effective potential in a finite volume $P(\phi)_2$ quantum field theory to the expected energy density under the constraint of a fixed average field. In the one phase region, i.e., where the classical potential equals its convex hull and has nonvanishing second derivative, it is shown via a central limit theorem that in the infinite volume limit the effective potential is equal to the constrained energy density, provided $\hbar$ is sufficiently small.

1. Introduction

The effective potential in a quantum field theory is the Legendre transform of the generator of the connected Feynman vacuum graphs. In [2] it is argued that the physical meaning of the effective potential evaluated at the classical field a is the expected energy density under the constraint that the average field has value a. In [6] a heuristic argument using functional integrals is given in support of this physical interpretation. Steps are taken towards making this interpretation rigorous in [9].

Since the constrained energy density is in general not convex in finite volume, while the effective potential is convex, the two quantities cannot in general be equal in finite volume. In Theorem 1 below, an explicit formula is given relating the effective potential of a finite volume $P(\phi)_2$ theory to the expected energy density under the constraint of a fixed average field. In Theorem 2 below, it is shown using a central limit theorem that in the infinite volume limit the two quantities are equal in the one phase region, provided $\hbar$ is sufficiently small.

We now introduce the notation. Let P be a polynomial of degree greater than or equal to four which is bounded below, and let $m>0$. For $\mu\in\mathbb{R}$, let

$$U_\mu(a)=P(a)+\tfrac{1}{2}m^2a^2-\mu a.$$

The classical potential of the model is then U_0. The one-phase region is the complement of the set B, defined as follows:

$$B=\{a\in\mathbb{R}\colon U_0(a)\neq(\operatorname{conv}U_0(a)\}^-\cup\{a\in\mathbb{R}\colon U_0''(a)=0\},$$

where $\operatorname{conv}U_0$ denotes the convex hull of U_0. Let $d\mu$ be the Gaussian measure on