Commun. Math. Phys. 104, 151–162 (1986)

Communications in
Mathematical
Physics
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A Conformal Lower Bound for the Smallest Eigenvalue of the Dirac Operator and Killing Spinors

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Abstract. On a Riemannian spin manifold, we give a lower bound for the square of the eigenvalues of the Dirac operator by the smallest eigenvalue of the conformal Laplacian (the Yamabe operator). We prove, in the limiting case, that the eigenspinor field is a killing spinor, i.e., parallel with respect to a natural connection. In particular, if the scalar curvature is positive, the eigenspinor field is annihilated by harmonic forms and the metric is Einstein.

1. Introduction

In 1963 Lichnerowicz [12] proved that on a Riemannian spin manifold the square of the Dirac operator $\mathscr{D}$ is given by

$$\mathscr{D}^2 = \Delta + \frac{s}{4}, \tag{1.1}$$

where Δ is the positive spinor Laplacian and s the scalar curvature. This formula implies

Theorem [12]. *On a compact Riemannian spin manifold (M, g) of positive scalar curvature,*

(i) *there is no non-zero harmonic spinor, and*

(ii) *any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 > \tfrac{1}{4} \inf_M s. \tag{1.2}$$

Part (i) together with the Atiyah-Singer index theorem applied to the Dirac operator for $4k$-dimensional manifolds, gives a topological obstruction – namely the vanishing of the $\hat{A}$-genus of Hirzebruch – for the existence of positive scalar curvature metrics on a compact spin manifold.

Hitchin [7] extended Lichnerowicz' result of the vanishing of the KO-characteristic numbers defined by Milnor [13]. By introducing the notion of enlargeable manifolds and combining it with the spin condition, Gromov and