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Can One Hear the Dimension of a Fractal?

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Dedicated to the memory of Mark Kac

Abstract. We consider the spectrum of the Laplacian in a bounded open domain of $\mathbb{R}^n$ with a rough boundary (i.e. with possibly non-integer dimension) and we discuss a conjecture by M. V. Berry generalizing Weyl's conjecture. Then using ideas Mark Kac developed in his famous study of the drum, we give upper and lower bounds for the second term of the expansion of the partition function. The main thesis of the paper is to show that the relevant measure of the roughness of the boundary should be based on Minkowski dimensions and on Minkowski measures rather than on Haussdorff ones.

1. Introduction

Let n be an integer, let D be a fixed bounded open set in $\mathbb{R}^n$, and let us denote by H_D (H when the domain in D is understood) minus one half the Dirichlet Laplacian in D. If we consider H_D as a self adjoint operator on $L^2(D, dx)$, its spectrum is discrete and can be written down in the form

$$0<\mu_1<\mu_2\leqq\mu_3\ldots\infty\,.$$

We consider the counting function

$$N_D(\mu)=\#\{j\geqq 1;\mu_j\leqq\mu\}\,.$$

With these notations, Weyl's famous theorem states that:

$$N_D(\mu)\sim c_n\mathfrak{L}_n(D)\mu^{n/2}\,,$$

as μ tends to ∞, where $\mathfrak{L}_n$ denotes the usual Lebesgue measure in $\mathbb{R}^n$, and where c_n is a universal constant depending only on the dimension n. Whenever the boundary of D is smooth, it was even proven that:

$$N_D(\mu)=c_n\mathfrak{L}_n(D)\mu^{n/2}+O(\mu^{(n-1)/2})\,,$$