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Evaluation of the One Loop String Path Integral

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Abstract. We evaluate Polyakov's path integral for the sum over all closed surfaces with the topology of a torus, in the critical dimension $d = 26$. The result is applied to the partition function and cosmological constant of the free bosonic string, and to tachyon scattering amplitudes.

In this paper, we evaluate the sum over all closed 2-surfaces with the topology of a torus. Our starting point is the path integral of Polyakov [1]:

$$W = \int \frac{dg_{ab}dx^\mu}{V_{GC}V_W} \exp\left(-\int d^2\sigma \sqrt{g}\left[\frac{T}{2} g^{ab}\partial_a x^\mu \partial_b x^\mu + \lambda R + \mu^2 \right]\right). \tag{1}$$

The integration runs over all Euclidean metrics $g_{ab}(\sigma)$ on a two-surface of given topology, and all embeddings $x^\mu(\sigma)$ of the 2-surface into d-dimensional Euclidean spacetime. T is the string tension. The λR term is proportional to the Euler number of the surface, and vanishes for the torus. The action is invariant under changes of the coordinates σ of the world sheet. Classically, when $\mu^2 = 0$, there is a second local symmetry group, the Weyl transformations

$$\delta g_{ab}(\sigma) = \lambda(\sigma) g_{ab}(\sigma). \tag{2}$$

This remains a symmetry of the quantum theory provided $d = 26$ and provided the counterterm μ^2 is appropriately chosen. We restrict our attention to this case, of exact Weyl invariance. The volumes of the local symmetry groups, V_{GC} and V_W respectively, must be divided out of the integration. We show that this can be done in an unambiguous way, leaving a finite measure; genus 1, the torus, is particularly simple in this respect. Our result is not new, since the one loop closed string graph has been evaluated by operator methods [2]. However, it is useful to obtain it directly from (1). Friedan [3] and Alvarez [4] have given general discussions of the sums over surfaces with handles. As a check, we calculate the free energy of a gas of free strings, and also compare with the expression for one loop string graphs as obtained by operator methods [2]. For the one loop cosmological constant of the free bosonic string theory we find a surprising result: it is not equal to the sum of the one loop contributions of the individual particles.

In order to describe a torus, we take x^μ and g_{ab} to be periodic functions of σ^1 and