Commun. Math. Phys. 103, 599–636 (1986)

Communications in Mathematical Physics

Lattice Gauge Fields, Principal Bundles and the Calculation of Topological Charge

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Abstract. SU_2-valued lattice gauge fields are studied on a 4-dimensional simplicial lattice. If **u** has sufficiently small plaquette products, then there is a unique principal SU_2-bundle ξ admitting transition functions, defined on the intersections of adjacent dual cells, which take values within $\pi/8$ of **u**. An algorithm is explicitly given which associates an integer to every **u** off a certain set of measure zero. This algorithm only involves evaluation of 4×4 determinants and the solution of quadratic equations. When **u** is as above, the integer produced is the second Chern number of ξ, i.e. the topological charge of **u**.

1. Introduction

This article has a theoretical and a practical side. It analyzes the circumstances under which an SU_2-valued lattice gauge field determines a principal SU_2-bundle, and it also presents an explicit algorithm which then computes the second Chern number of that bundle, the *topological charge*, directly from the lattice data. This algorithm, which involves no more than 4×4 determinants and the solution of quadratic equations, has been used (in joint work with Gordon Lasher) for a Monte Carlo calculation of topological susceptibility; the details and results are reported elsewhere [15].

For early work on this problem, see [5, 6], and also [13, 17]; our work grew out of an attempt to find an appropriate mathematical context for Martin Lüscher's construction [18]. Lüscher's algorithm has recently been programmed [9]; other topological charge algorithms have been given in [14, 24, 30, 31], and have been discussed in [21, 22].

Lattice gauge fields were introduced by Kenneth Wilson in 1974 [29] (see also [25]) to represent classical field configurations in Monte Carlo evaluations of path integral solutions of quantum field theories. Here is the context. (We shall assume for simplicity in this work that we are dealing with a compact space-time X; as usual, the time coordinate has been rotated in the complex plane to give X a Euclidean metric.) Let us fix a compact Lie group G. The set of gauge fields on X