Commun. Math. Phys. 103, 177–240 (1986)

Communications in Mathematical Physics

Differential Equations in the Spectral Parameter *

J. J. Duistermaat[1] and F. A. Grünbaum[2]

[1] Mathematisch Instituut der Rijksuniversiteit Utrecht, The Netherlands
[2] Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract. We determine all the potentials $V(x)$ for the Schrödinger equation $(-\partial_x^2+V(x))\phi=k^2\phi$ such that some family of eigenfunctions ϕ satisfies a differential equation in the spectral parameter k of the form $B(k,\partial_k)\phi=\Theta(x)\phi$. For each such $V(x)$ we determine the algebra of all possible operators B and the corresponding functions $\Theta(x)$.

Table of Contents

0. Introduction

In this paper we study the following question: For which linear ordinary differential operators $L=\sum_{j=0}^{l} L_j(x)\cdot\left(\frac{\partial}{\partial x}\right)^j$ is there a non-zero family of eigenfunctions $\phi(x,\lambda)$,

$$(L\phi)(x,\lambda)=\lambda\cdot\phi(x,\lambda)\,, \tag{0.1}$$

depending smoothly on the eigenfunction parameter λ, which is *also* an eigenfunction of a linear ordinary differential operator $A=\sum_{r=0}^{m} A_r(\lambda)\cdot\left(\frac{\partial}{\partial\lambda}\right)^r$

$$(A\phi)(x,\lambda)=\Theta(x)\cdot\phi(x,\lambda) \tag{0.2}$$

for an eigenvalue Θ which is a function of x?

* This research was partially supported by NSF grant DMS 84-03232 and ONR contract NOOO14-84-C-0159