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Field Algebras do not Leave Field Domains Invariant

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Abstract. It is proved that von Neumann algebras associated to Op*-algebra $(\mathscr{P}, \mathscr{D})$ cannot leave the domain $\mathscr{D}$ of $\mathscr{P}$ invariant if they are type I or type III factors or finite direct sums of such factors. Hence it follows that in quantum field theory global and local von Neumann field algebras in typical cases do not leave invariant the definition domain of Wightman fields.

In the theory of Op*-algebras it is often necessary or just helpful to consider von Neumann algebras associated in various ways to a given Op*-algebra $(\mathscr{P}, \mathscr{D})$. E.g., one of such algebras is $(\mathscr{P}'_w)'$, where $\mathscr{P}'_w$ is the weak bounded commutant of $\mathscr{P}$, and there are many others (cf. [1, Sect. 8.1]) which are usually larger because $\mathscr{P}'_w$ is the largest of all bounded commutants of $\mathscr{P}$. We are going to demonstrate now that there is strong connection between the structure of these von Neumann algebras and their property to leave or not to leave invariant the domain $\mathscr{D}$ of $\mathscr{P}$. The result is in itself quite simple but it has interesting implications for quantum field theory: it turns out that field W^*-algebras, global as well as local, under rather general conditions cannot leave invariant the definition domain of field operators (Gårding domain of Wightman theory). We also get some restrictions on admissible classes of field Op*-algebras.

We call Op*-algebra $(\mathscr{P}, \mathscr{D})$ nontrivial if it does not coincide with its bounded part, $\mathscr{P} \neq \mathscr{P}_b = \mathscr{P} \cap \mathfrak{B}(\mathscr{H})$, and we denote the involution in $\mathscr{P}$ as $A^\dagger = A^*|_{\mathscr{D}}$, $A \in \mathscr{P}$, and the strong bounded commutant of $\mathscr{P}$ as $\mathscr{P}'_s$. Our main statement is the following

Theorem. *Let $(\mathscr{P}, \mathscr{D})$ be a nontrivial* Op*-*algebra such that $\mathscr{P}'_s = \mathscr{P}'_w$ or including at least one essentially self-adjoint unbounded operator, and let R be a von Neumann algebra such that $R\mathscr{D} \subset \mathscr{D}$ and $R \supset (\mathscr{P}'_w)'$. Then R is not a type I or type III factor. Moreover, if $\mathscr{P}'_s = \mathscr{P}'_w$ or $\mathscr{P}$ is generated by a system of essentially self-adjoint operators, then R is not a finite direct sum of type I and/or type III factors.*

Proof. Let us show that R includes spectral projections of a self-adjoint extension $\tilde{A}$ of some unbounded operator $A \in \mathscr{P}$. In the case $\mathscr{P} \ni A$, $\bar{A} = A^*$, $A \notin \mathfrak{B}(\mathscr{H})$, let us denote $E(\lambda)$ the spectral measure of $\tilde{A} \equiv \bar{A}$. It is easy to check that any operator $B \in \mathscr{P}'_w$ commutes weakly with $\bar{A}$ and leaves its domain invariant. Hence, B commutes strongly with $\bar{A}$ and, by the spectral theorem, commutes with $E(\lambda)$, so we get $E(\lambda) \in (\mathscr{P}'_w)'$. In the general case, for $A_1 \in \mathscr{P}$, $A_1 \notin \mathfrak{B}(\mathscr{H})$, let us denote $\tilde{A}$ the Friedrichs' extension of the positive unbounded operator $A = A_1^\dagger A_1 \in \mathscr{P}$ and $E(\lambda)$ the spectral measure of $\tilde{A}$. A unitary operator commuting strongly with A commutes strongly with $\tilde{A}$ [2, p. 358] and therefore it commutes with $E(\lambda)$. Hence