

Signs of the Ising Model Ursell Functions

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Abstract. It is proven that the Ursell functions U_{2k} of the Ising model have the conjectured signs: $(-1)^{k+1}U_{2k} \geq 0$. The proof is based on Aizenman’s random current representation and combinatorics.

1. Introduction

The Ursell function $U_k(\sigma_1, \dots, \sigma_k)$ of a family of k random variables $\sigma_1, \dots, \sigma_k$ is defined by means of a generating function:

$$U_k(\sigma_1, \dots, \sigma_k) = \frac{\partial^k}{\partial h_1 \dots \partial h_k} \ln \left\langle \exp \sum_{i=1}^k h_i \sigma_i \right\rangle \Big|_{h=0}. \tag{1}$$

Here $\langle \rangle$ stands for expectation. Another way to define them is by the formula

$$U_k(\sigma_1, \dots, \sigma_k) = \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|-1} (|\mathcal{P}|-1)! \sum_{P \in \mathcal{P}} \left\langle \prod_{p \in P} \sigma_p \right\rangle, \tag{2}$$

where the summation is over all partitions \mathcal{P} of the set $I = \{1, \dots, k\}$, $\mathcal{P} = \{P_1, \dots, P_r\}$, $|\mathcal{P}| = r$, $\bigcup_{i=1}^r P_i = I$, $P_i \cap P_j = \emptyset$, $i \neq j$. The formula (2) follows from (1) by a straightforward calculation.

In this paper we study the Ursell functions of the general Ising ferromagnet with pair interaction. We have a collection of random variables $\sigma_1, \dots, \sigma_N$, $\sigma_i = \pm 1$, whose joint distribution is given by the probabilities

$$P_N(\sigma) = Z^{-1} \exp \left\{ \sum_{s,t=1, s \neq t}^N J_{st} \sigma_s \sigma_t \right\}, \tag{3}$$

where the partition function

$$Z = \sum_{\sigma} \exp \left\{ \sum_{s,t=1, s \neq t}^N J_{st} \sigma_s \sigma_t \right\}, \tag{4}$$

and $J_{st} \geq 0$ for all $s, t = 1, \dots, N$.