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The Loop Expansion for the Effective Potential in the $P(\phi)_2$ Quantum Field Theory*

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Abstract. We study the loop expansion for the effective potential, defined as the Fenchel transform (convex conjugate) of the pressure in an external field, in the $P(\phi)_2$ quantum field theory. For values of the classical field a for which the classical potential $U_0(a)=P(a)+\frac{1}{2}m^2a^2$ equals its convex hull and has nonvanishing curvature we prove that the 1-PI loop expansion is asymptotic as $\hbar\downarrow 0$. We also give an example of a double well classical potential for which the 1-PI loop expansion fails to be asymptotic, and find the true asymptotics.

1. Introduction

The effective potential for the $P(\phi)_2$ Euclidean quantum field theory is defined as the Fenchel transform of the pressure in an external field:

$$V(\hbar,a)=\sup_{\mu\in\mathbb{R}}[\mu a-p(\hbar,\mu)]\,. \tag{1.1}$$

Here the positive parameter $\hbar$ is Planck's constant divided by 2π, the classical field a is real, and $p(\hbar,\mu)$ is given by

$$p(\hbar,\mu)=\hbar\lim_{\Lambda\uparrow\mathbb{R}^2}\frac{1}{|\Lambda|}\ln\int\exp\left[\frac{-1}{\hbar}\int_{\Lambda}[:P(\phi(x)):-\mu\phi(x)]\,dx\right]d\mu_{\hbar C}\,, \tag{1.2}$$

where $C=(-\Delta+m^2)^{-1}$ for some $m^2>0$, $d\mu_{\hbar C}$ is Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with covariance $\hbar C$, the Wick order is with respect to $\hbar C$, and $\Lambda\uparrow\mathbb{R}^2$ through a sequence of rectangles. In [14] the limit (1.2) is shown to exist for a wide variety of boundary conditions on $\partial\Lambda$, in particular for periodic boundary conditions which we will use unless otherwise indicated.

The importance of the effective potential is that it characterizes the occurrence of phase transitions in the theory [2, 16]: linear portions of $V(\hbar,\cdot)$ are in a one-

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