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The Variational Problem and Background Fields in Renormalization Group Method for Lattice Gauge Theories*

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Abstract. We consider the action of a lattice gauge theory on a space of regular gauge field configurations with fixed averages, and we prove that there exists a minimum of this action. The minimum is unique up to gauge transformations. This minimal configuration is called a background field, and it serves as a basis of an expansion and perturbative methods.

It was explained in [1] that the fundamental step in our renormalization group approach is to find solutions of the variational problem and to investigate their regularity properties and expansions. Let us state the problem precisely. To formulate it we recall some definitions introduced in [3, 4, 6]. This paper is based on the results of those papers, and we refer the reader to them for more detailed explanations of the definitions and the results.

At first let us recall the geometric setting. We assume that a sequence of domains Ω_j, $j=0, 1,\dots,k$, is given, satisfying the following conditions: $\Omega_j \subset T_\eta$, $\Omega_0 \supset \Omega_1 \supset \dots \supset \Omega_k$, Ω_j is a union of big block of the size $M_1 L^j \eta$,

$$(L^j\eta)^{-1}\,\mathrm{dist}(\Omega_j^c, \Omega_{j+1}) > RM_1, \tag{1}$$

where $R \geqq R_1$, the numbers R_1, M_1 are fixed in such a way that all the results of [3, 5, 6] hold for these numbers. We identify domains Ω_j with sets of bonds or plaquettes in the usual way, as sets of bonds with at least one end-point belonging to Ω_j, or sets of plaquettes with at least one corner belonging to Ω_j. This remark applies to other sets also. The sets Λ_j and $\mathfrak{B}_k$ are defined as

$$\Lambda_j = \Omega_j^{(j)} \setminus \Omega_{j+1}^{(j)}, \quad \Omega_{k+1} = \emptyset, \quad \text{or} \quad B^j(\Lambda_j) = \Omega_j \setminus \Omega_{j+1},$$
$$\mathfrak{B}_k = \bigcup_{j=0}^{k} \Lambda_j .$$

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