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Curvature and the Evolution of Fronts

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Abstract. The evolution of a front propagating along its normal vector field with speed F dependent on curvature K is considered. The change in total variation of the propagating front is shown to depend only on dF/dK only where K changes sign. Analysis of the case $F(K)=1-\varepsilon K$, where ε is a constant, shows that curvature plays a role similar to that of viscosity in Burgers equation. For $\varepsilon=0$ and non-convex initial data, the curvature blows up, corners develop, and an entropy condition can be formulated to provide an explicit construction for a weak solution beyond the singularity. We then numerically show that the solution as ε goes to zero converges to the constructed weak solution. Numerical methods based on finite difference schemes for marker particles along the front are shown to be unstable in regions where the curvature builds. As a remedy, we show that front tracking based on volume of fluid techniques can be used together with the entropy condition to provide transition from the classical to weak solution.

I. Introduction

We study the evolution of a front propagating along its normal vector field with speed a function of curvature. We first prove a general result relating the growth/decay of the total variation to the speed. We then study the case of a front moving with speed $1-\varepsilon K$, where ε is a constant and K is the curvature, and show that the curvature term plays a smoothing role in the solution similar to that of viscosity in Burgers equation. For $\varepsilon=0$, in which case the front moves at constant speed, the curvature blows up, differentiability is lost, and an entropy condition can be formulated to provide an explicit construction of a weak solution beyond the singularity. For $\varepsilon>0$, we then solve the equations of motion numerically and show that in the limit as ε goes to zero, the solution converges to our constructed weak solution. We show that corners which develop in the propagating front

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