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Period Doubling Cascades of Attractors: A Prerequisite for Horseshoes

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In memory of Charles Conley

Abstract. This paper shows that if a horseshoe is created in a natural manner as a parameter is varied, then the process of creation involves the appearance of attracting periodic orbits of all periods. Furthermore, each of these orbits will period double repeatedly, with those periods going to infinity.

1. Introduction

A two-dimensional horseshoe map is defined on a neighborhood of a rectangle and maps it into a horseshoe shape that lies across the original rectangle, as in Fig. 1c. An analogous map can be defined in higher dimensions. Figure 1f shows the three dimensional case. Since Smale described horseshoes [S], many dynamical systems have been shown to have them. In the fully developed horseshoe, as described by Smale, there are necessarily immense numbers of periodic orbits: the k^{th} iterate of the map has 2^k fixed points. The dynamical behavior in the rectangle is indeed complicated; however, all the periodic orbits there are unstable, and almost any initial point chosen in the rectangle has a trajectory that eventually leaves it.

This paper investigates the dynamical behavior as the horseshoe is being formed in the typical way shown in Fig. 1. At the initial parameter value ($\lambda = 0$), the rectangle and its image are disjoint; at $\lambda = 1$, there is a horseshoe map, with infinitely many unstable periodic points. We assume that for each parameter value,

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