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A Solution to the Navier–Stokes Inequality with an Internal Singularity

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Abstract. We consider weak solutions to the time dependent Navier–Stokes equations of incompressible fluid flow in three dimensional space with an external force that always acts against the direction of the flow. We show that there exists a solution with an internal singularity. The speed of the flow reaches infinity at this singular point. In addition, the solution has finite kinetic energy.

Section 1. Introduction

The purpose of this paper is to prove Theorem 1.1 below. The statement of this theorem is followed by an informal explanation of what it says.

Definition. If f is a function defined on an open subset of $R^3 \times R$, then the laplacian and the gradient of f will involve only the R^3 variables. Thus,

$$\Delta f(x,t) = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}(x,t) \quad \text{and} \quad \nabla f(x,t) = \left(\frac{\partial f}{\partial x_1}(x,t), \frac{\partial f}{\partial x_2}(x,t), \frac{\partial f}{\partial x_3}(x,t) \right).$$

The norm $|f|$ will always be the euclidean norm. For example, in (1.7) we have $|\nabla u|^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2$.

Theorem 1.1. *There exist functions* $u: R^3 \times [0, \infty) \to R^3$ *and* $p: R^3 \times [0, \infty) \to R$ *with the following properties:*

$$\text{there is a compact set } K \subset R^3 \text{ such that } u(x,t) = 0 \quad \text{for all } x \notin K, \tag{1.1}$$

$$\text{for fixed } t, \text{ the function } u_t: R^3 \to R^3 \text{ defined by } u_t(x) = u(x,t) \text{ is a } C^\infty \text{ function}, \tag{1.2}$$

$$\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}(x,t) = 0, \tag{1.3}$$

$$p(x,t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}(y,t) \frac{\partial u_i}{\partial x_j}(y,t)(4\pi|x-y|)^{-1} dy, \tag{1.4}$$

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