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Hyperbolicity, Sinks and Measure in One Dimensional Dynamics

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Abstract. Let f be a C^2 map of the circle or the interval and let $\Sigma(f)$ denote the complement of the basins of attraction of the attracting periodic orbits. We prove that $\Sigma(f)$ is a hyperbolic expanding set if (and obviously only if) every periodic point is hyperbolic and $\Sigma(f)$ doesn't contain the critical point. This is the real one dimensional version of Fatou's hyperbolicity criteria for holomorphic endomorphisms of the Riemann sphere. We also explore other applications of the techniques used for the result above, proving, for instance, that for every C^2 immersion f of the circle (i.e. a map of the circle onto itself without critical points), either its Julia set has measure zero or it is the whole circle and then f is ergodic, i.e. positively invariant Borel sets have zero or full measure.

Introduction

The subject of this paper is the dynamics of C^2 maps of the circle or the interval, on regions bounded away from the critical points. The aspects of the dynamics that we shall consider, and the corresponding results that we shall prove, can be summarized as follows:

Hyperbolicity—If the map is not topologically equivalent to an irrational rotation of the circle, every compact invariant set not containing critical points, sinks or non-hyperbolic periodic points is hyperbolic.

Stability—Structural stability is generic in the space of C^r immersions of the circle and is characterized by the hyperbolicity of the non-wandering set.

Ergodicity—Transitive C^2 immersions of the circle are ergodic, i.e. every invariant Borel set has either zero or full Lebesgue measure.

Measure—If Γ is a compact invariant set with empty interior not containing critical points, then either the Lebesgue measure of Γ is zero or there exists an interval U that is mapped diffeomorphically into itself by some power of the map and such that $\Gamma \cap U$ has positive Lebesgue measure.

Sinks—For every compact set K that doesn't contain critical points, the periods of the sinks or non-hyperbolic periodic orbits contained in K are bounded.