Commun. Math. Phys. 100, 439–446 (1985)

Communications in
Mathematical
Physics

The Reciprocal of a Borel Summable Function is Borel Summable

G. Auberson and G. Mennessier

Departement de Physique Mathématique*, U.S.T.L., F-34060 Montpellier Cedex, France

Abstract. It is proven that if a function f is Borel summable in some angular region and has a non-vanishing derivative at the origin, then its reciprocal f^{-1} is also Borel summable in a region which has essentially the same angular extent.

Formal manipulations of divergent (or presumably divergent) power series are frequently done when non-exactly solvable problems are treated through the perturbation method, especially in quantum mechanics and in quantum field theory. In some favourable circumstances, i.e. when the particular quantity to be expanded perturbatively turns out to be a Borel summable function, these manipulations can be justified by appealing to general properties of such functions. Admittedly, for the most interesting cases, e.g. for the non-abelian gauge theories, it is very unlikely that the Green functions (say) enjoy the required Borel summability property [1]. In this context, it has been argued however [2], that a proper use of renormalization group methods (especially through the freedom in the choice of the renormalization scheme) might improve the situation, at least for the quantities of physical interest, so that the issue seems (at least to us!) still inconclusive[1].

In any case, we believe that it is interesting to gather as many results as possible about the general properties of Borel summable functions, which potentially may give a firm basis to the above mentioned formal manipulations. This was precisely the purpose of our work in [3]. In that paper however, an aspect of Borel summability was not touched upon, namely the problem of inverting a Borel summable function, which may be of some relevance in the renormalization

---

* Physique Mathématique et Théorique, Unité associée au CNRS No. 768

1 To avoid a possible confusion, let us notice that we are using the expression "Borel summable" in the full mathematical sense (and not simply to mean that the perturbation *series* is formally Borel summable). This convention is not shared by all authors (for instance an explicit distinction is made by Stevenson [2] who calls "Borel recoverable" a "Borel summable" function in our acceptation)