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Yang-Mills Equations on the Two-Dimensional Sphere

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Abstract. We construct a 1 : 1 correspondence between the equivalence classes of Yang-Mills fields over S^2 and the conjugacy classes of closed geodesics of the structure group. Furthermore, we give an explicit isolation theorem for any Yang-Mills field over S^2.

Let G be a compact and connected Lie group and consider a principal fibre bundle (P, M, G) over a compact oriented Riemannian manifold M. On the space $\mathscr{C}(P)$ of connections on P the Yang-Mills functional $L: \mathscr{C}(P) \to R$ is defined by

$$L(Z) = \int_M \|\Omega_Z\|^2 dM,$$

where Ω_Z is the curvature of the connection Z and the norm is defined by the Riemannian metric on M and a fixed Ad-invariant scalar product on the Lie algebra of G. The critical points of the Yang-Mills functional are called Yang-Mills connections. It is well-known (see [3]) that $Z \in \mathscr{C}(P)$ is a Yang-Mills connection if and only if Z is a solution of the Yang-Mills equation $D_Z * \Omega_Z = 0$. In the introduction of [2] Atiyah and Bott mentioned that, for the 2-sphere S^2, the Yang-Mills equation for G essentially reproduced the Morse theory picture for the loop space ΩG. Similar ideas can be found in papers concerning magnetic monopoles (see [4, 6, 10]). The purpose of the present paper is to work out this idea exactly and to complete it.

In the first section we investigate a mapping $\Phi: \mathscr{C}(P) \to \Omega G$ (see [4, 12]) from the space of connections on a principal fibre bundle (P, S^2, G) onto a subspace of ΩG determined by the topological type of P.

In Sect. 2 we show that the mapping Φ carries the critical points of the Yang-Mills functional into the critical points of the energy integral on ΩG. For this we prove that the holonomy group of a Yang-Mills connection over S^2 is either trivial or the group S^1. Furthermore, calculating the index and the nullity of a Yang-Mills connection over S^2 by means of cohomology theory and studying the Jacobi equation on the corresponding closed geodesic, we verify that, after identification