Commun. Math. Phys. 99, 593–612 (1985)

Communications in Mathematical Physics

Are There Critical Points on the Boundaries of Singular Domains?

Michael R. Herman

Centre de Mathématiques, Ecole polytechnique, F-91128 Palaiseau Cedex, France

Abstract. Generalizing a result of E. Ghys, we prove a general theorem that implies that if a rational function f of the Riemann sphere of degree $\geqq 2$ leaves invariant a singular domain C (a disk or a ring) on which the rotation number of f satisfies a diophantine condition, provided that on $\bar{C}$ f is injective, then each boundary component of C contains critical point of f. The injectivity condition is always satisfied for singular disks associated to linearizable periodic elliptic points of $f(z)=z^n+a$, with $n\in\mathbb{N}$, $n\geqq 2$ and $a\in\mathbb{C}$. We also show that the singular disks, associated to periodic elliptic points of $f(z)=e^{az}$ that satisfy a diophantine condition, are unbounded in $\mathbb{C}$. In the end of the paper, we give a survey of the theory of iteration of entire functions of $\mathbb{C}$.

Introduction

In 1920, Fatou [3, § 30] showed that if a rational function g, of degree $\geqq 2$, left invariant a singular domain C, then the frontier (or boundary) of C in $\mathbb{S}^2$ was contained in the ω-limit set under g of the set critical points of g (C is either $\mathbb{C}$-diffeomorphic to $\mathbb{D}$ or to a ring). In 1942, Siegel [9] showed the existence of singular domains that are disks and for the existence of rings we refer to [6, I and VIII].

We propose to show in II.1, under the restrictions that the rotation number α of $g_{|C}$ satisfies a diophantine condition and $g_{|\bar{C}}$ is injective, then g has at least one critical point in each component of $\mathrm{Fr}(C)$. This generalizes the partial result obtained by Ghys [4] who supposed that $\mathrm{Fr}(C)$ was a Jordan curve.

This result follows immediately from the main theorem (I.4). The proof of the main theorem was influenced by work of Ghys [4] who introduced diffeomorphisms of the circle, by using a form of conformal welding for special cases of Theorem 1 and related questions. The main ingredient to prove Theorem 1 is the fundamental theorem of [5, IX], as generalized by Yoccoz [10] (we use [10] to obtain rotation numbers that satisfy a diophantine condition and not the "unnatural" set of numbers of [5] that satisfy a condition A). The exact arithmetic for which the theorem of I.4 is true is an open question (see I.3).