Commun. Math. Phys. 99, 365–384 (1985)

Communications in Mathematical Physics

Non-Compact Symmetric Spaces and the Toda Molecule Equations

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Abstract. It has been shown by Olshanetsky and Perelomov that the Toda molecule equations associated with any Lie group G describe special geodesic motions on the Riemannian non-compact symmetric space which is the quotient of the normal real form of G, G^N, by its maximal compact subgroup. This is explained in more detail and it is shown that the "fundamental Poisson bracket relation" involving the Lax operator A and leading to the Yang–Baxter equation and integrability properties is a direct consequence of the fact that the Iwasawa decomposition for G^N endows the symmetric space with a hidden group theoretic structure.

1. Introduction

Integrable systems are currently of great interest for a variety of reasons, mathematical and physical [1, 2]. One reason is that some four dimensional gauge theories, perhaps spontaneously broken by a Higgs mechanism may belong to a new class of integrable theory. The integrability may relate to the electromagnetic duality conjectures [3] whose validity is most favoured in the $N = 4$ supersymmetric gauge theories [4]. It is known that the radial dependence of certain spherically symmetric monopole solutions occurring in such theories is governed by a Toda molecule equation [5, 6] (with t replaced by i times radius):

$$\frac{d^2}{dt^2}\phi_a = -\exp\left(\sum_b K_{ab}\phi_b\right), \quad a = 1,2,..r. \tag{1.1}$$

These equations are known to be completely integrable when the square, non-singular, matrix K is the Cartan matrix for a simple Lie group G (of rank r). The solutions to these equations are also known [7, 8, 9, 6].

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* Supported by CNP_q (Brasil)