Commun. Math. Phys. 99, 319–345 (1985)

Communications in Mathematical Physics

The Hamiltonian Structure of General Relativistic Perfect Fluids

David Bao[1]*, Jerrold Marsden[2]** and Ronald Walton[3]

1 School of Mathematics, The Institute for Advanced Study, Princeton, NJ 08540, USA and Department of Mathematics, University of Houston-University Park, Houston, TX 77004, USA

2 Department of Mathematics, University of California, Berkeley, CA 94720, USA

3 Space Systems Division, Lockheed Missiles & Space Company, Inc., 1111 Lockheed Way, Sunnyvale, CA 94086, USA

Abstract. We show that the evolution equations for a perfect fluid coupled to general relativity in a general lapse and shift, are Hamiltonian relative to a certain Poisson structure. For the fluid variables, a Lie-Poisson structure associated to the dual of a semi-direct product Lie algebra is used, while the bracket for the gravitational variables has the usual canonical symplectic structure. The evolution is governed by a Hamiltonian which is equivalent to that obtained from a canonical analysis. The relationship of our Hamiltonian structure with other approaches in the literature, such as Clebsch potentials, Lagrangian to Eulerian transformations, and its use in clarifying linearization stability, are discussed.

Introduction

The purpose of this paper is to study the Hamiltonian nature of the Einstein–Euler field equations, which govern the "Eulerian" description of general relativistic perfect fluids. We show that the corresponding evolution equations, when written in a general lapse and shift, are Hamiltonian with respect to a Poisson bracket given by the Dirac-ADM canonical Poisson bracket for the gravitational variables (Dirac, 1959, Arnowitt et al., 1962) and by a "Lie-Poisson" bracket for the fluid variables.

A Lie-Poisson bracket is a noncanonical Poisson bracket defined by a Lie algebraic structure associated with the symmetries of a Hamiltonian system. For non-relativistic fluids, these brackets are due to Morrison and Greene (1980) and Dzyaloshinski and Volovik (1980). The group theoretic origin of Lie-Poisson brackets for Hamiltonian systems and their derivation from canonical brackets in

* Research supported in part by NSF grant MCS 81-08814(A02)

** Research supported in part by NSF grant MCS 81-07086