Commun. Math. Phys. 99, 177–195 (1985)

Communications in
Mathematical
Physics

On the Concept of Attractor

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Abstract. This note proposes a definition for the concept of "attractor," based on the probable asymptotic behavior of orbits. The definition is sufficiently broad so that every smooth compact dynamical system has at least one attractor.

Attractors have played an increasingly important role in thinking about dynamical systems since their introduction some twenty years ago; yet there is no agreement as to the most useful definition. Section 1 of this note compares several definitions from the literature, and Sect. 2 proposes an alternative definition based on asymptotic behavior for almost every choice of initial point. The remaining two sections illustrate this definition by a number of examples, and discuss the stability and robustness of these attractors. There are three appendices. The first compares a closely related purely topological definition, the second studies real quadratic maps of their interval as a test case, and the third discusses strange attractors.

1. History

The following is quoted from Auslander, Bhatia, and Seibert (1964):

> "In the study of topological properties of ordinary differential equations, the stability theory of compact invariant sets (which may be regarded as generalizations of critical points and limit cycles) plays a central role. ... By Liapunov stability (or just stability) of the compact invariant set M, we mean that every orbit starting sufficiently close to M will remain in a given neighborhood of M. The set M is asymptotically stable if it is stable and is also an 'attractor' – that is, all orbits in a neighborhood ... of M approach M."

(Compare La Salle and Lefschetz, p. 31.) The word attractor, applied to a single invariant point for a smooth flow, had been used earlier by Coddington and