Commun. Math. Phys. 98, 525–537 (1985)

Communications in Mathematical Physics

Solution of the Initial Value Problem for the sine-Gordon Equation Using a Kac-Moody Algebra

Paul Mansfield*

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

Abstract. We solve the classical sine-Gordon equation using a Lax pair belonging to a Kac-Moody algebra. By realising the algebra in terms of fermionic currents we reduce the initial value problem to the evaluation of a fermionic propagator as a sum of Feynman diagrams.

1. Introduction

The sine-Gordon equation,

$$\frac{\partial^2 \varrho}{\partial x^2} - \frac{\partial^2 \varrho}{\partial t^2} = \sin \varrho ,$$

is a relativistic equation in one space and one time dimension with many applications in mathematical physics [1]. Soliton solutions have been constructed using the inverse scattering transform [2]. Here we describe a different approach, based on a Kac-Moody algebra, which solves the equation for arbitrary initial data. The method reveals an intimate relationship between solutions of the sine-Gordon equation and the propagator for a certain fermionic quantum field theory. This is a precursor at the classical level of the well-known equivalence between the quantum sine-Gordon theory and the massive Thirring model [3].

The starting point for the solution of the sine-Gordon equation by the inverse scattering transform is its associated Lax pair. Conventionally, this is constructed from elements of the Lie algebra $\mathrm{sl}(2, \mathbb{C})$. Leznov and Smirnov [4] have pointed out that the Kac-Moody algebra may be used instead. The advantage of this is that the same construction that was used to solve the Toda equations by Leznov and Saveliev [5] may be exploited to relate solutions of the sine-Gordon equation to elements of the Kac-Moody group constructed from initial data given on some light-cone. Jimbo and Miwa have also constructed solutions using the Kac-Moody algebra [8]. In Sect. 2 we give a different derivation of this construction

* Address from 1 October 1984: CERN, Geneva