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Remarks on a Paper by J. T. Beale, T. Kato, and A. Majda

Gustavo Ponce

Escuela de Fisica, Mathematicas, Fac, de Ciencias Universidad Central de Venezuela, Apartado 50.925 Caracas, Venezuela

Abstract. We prove that the maximum norm of the deformation tensor controls the possible breakdown of smooth solutions for the 3-dimensional Euler equations. More precisely, the loss of regularity in a local smooth solution of the Euler equations implies the growth without bound of the deformation tensor as the critical time approaches; equivalently, if the deformation tensor remains bounded the existence of a smooth solution is guaranteed.

The motion of an ideal incompressible fluid is described by a system of partial differential equations known as Euler equations. In [1] J. T. Beale, T. Kato, and A. Majda have given a mathematically rigorous connection between the accumulation of vorticity and the development of singularities for the three-dimensional Euler equations. In fact, they have shown that the maximum norm of the vorticity alone controls the breakdown of smooth solution of these equations. Thus one may ask: Does the blow up of the solution imply also the blow up of the deformation tensor in the maximum norm? or, may it stay bounded for a longer time? In this note we answer these questions. More precisely, we obtain the same results as those in [1], when the vorticity is substituted by the deformation tensor.

Thus we consider the system

$$\begin{array}{ll}\text{(a)} \\ \text{(b)}\end{array} \left\{\begin{array}{ll} u_t^k + u^j \cdot \partial_j u^k + \partial_k p = 0 & k = 1, 2, 3 \\ \operatorname{div} u = 0 & \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^3$, $t > 0$, $u = u(x,t) = (u^1, u^2, u^3)$ is the velocity field, and $p = p(x,t)$ is the pressure.

For this system the following local existence theorem is known: Given an initial velocity $u_0 \in H^s$, s integer, $s \geqq 3$ and $\operatorname{div} u_0 = 0$, there exists $T_0 = T_0(\|u_0\|_s)$ so that the system (1) has a unique solution $u \in C([0,T]: H^s) \cap C^1([0,T]: H^{s-1})$ at least for $T = T_0$. (See reference in [1]). □

Here we denote by $H^s = H^s(\mathbb{R}^3)$ (s being a positive integer) the Sobolev space consisting of functions whose distributional derivatives up to order s belong to $L^2(\mathbb{R}^3)$, and by $\|u\|_s$ the norm of u in H^s. Also, we use $\omega = \nabla \times u$ for the vorticity and $T = (T_{ij})$ $i, j = 1, 2, 3$, where $T_{ij} = \partial_j u^i + \partial_i u^j$ for the deformation tensor.